

On the self-consistency of the excursion set approach

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The excursion set approach provides a framework for predicting how the abundance of dark matter halos depends on the initial conditions. A key ingredient of this formalism comes from the physics of halo formation: the specification of a critical overdensity threshold (barrier) which protohalos must exceed if they are to form bound virialized halos at a later time. Another ingredient is statistical, as it requires the specification of the appropriate statistical ensemble over which to average when making predictions. The excursion set approach explicitly averages over all initial positions, thus implicitly assuming that the appropriate ensemble is that associated with randomly chosen positions in space, rather than special positions such as peaks of the initial density field. Since halos are known to collapse around special positions, it is not clear that the physical and statistical assumptions which underlie the excursion set approach are self-consistent. We argue that they are at least for low mass halos, and illustrate by comparing our excursion set predictions with numerical data from the DEUS simulations.

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The excursion set approach [1–3] is the most widely developed formalism for estimating how halo abundances depend on the background cosmology. Rather than directly predicting the comoving density of halos dn/dM in the mass range $[M, M + dM]$, it provides an estimate of the mass fraction in halos, $f(M) = (M/\rho) dn/dM$, where ρ is the comoving background density. That is to say, the approach seeks to answer the question: What is the probability that a randomly chosen particle in a simulation is in a halo of mass M ?

Building upon the seminal work of Press & Schechter [4], the excursion set assumes that at any random point in space the initial overdensity δ performs a random walk as function of the smoothing scale R . Since initial fluctuations are small, this scale can be associated to a mass, $M = \rho V(R)$, where $V(R)$ is the filtered volume. Then, the multiplicity function $f(M)$ is inferred from ensemble averaging over the number of trajectories that first-cross a non-linear collapse threshold (absorbing barrier). This condition solves the so called “cloud-in-cloud” problem due to miscounting the number of regions which are above the threshold at multiple scales.

The ensemble properties of the walks are determined by the statistics of the initial density fluctuation field and the filter function, which specify how the variance of the walk height depends on R and hence on mass: $\langle \delta^2 \rangle \equiv \sigma^2 \equiv S$ where $S = \frac{1}{2\pi^2} \int dk k^2 P(k) \tilde{W}^2(k, R)$. For a sharp-k filter, $\tilde{W}(k, R) = \theta(1/R - k)$ and $V(R) = 6\pi^2 R^3$,

it can be shown that random walks are uncorrelated, so the computation of the multiplicity function reduces to solving a simple diffusion problem [1]. In contrast, the top-hat filter in real space (sharp-x), $W(x, R) = \theta(x - R)$ and $V(R) = (4\pi/3) R^3$, observationally used to probe the linear density field, leads to correlated steps for which the computation of $f(M)$ is more challenging. In this case the effects of correlated steps can be accounted-for perturbatively using a path-integral approach [5–7] or other approximations [8–10].

Another key ingredient concerns the physics of collapse. This defines a critical density which must be exceeded if an object is to pull itself together against the expanding background. The simplest excursion set formulae assume a constant threshold as predicted by the spherical collapse model [11]. However, initial fluctuations are non-spherical and their evolution depends on the surrounding shear field [12], so the ellipsoidal collapse model [13] is assumed to be more realistic. This naturally gives rise to an approach with moving [14, 15] and/or stochastic [14–16] critical collapse barriers. The excursion set approach implicitly assumes that the statistics of patches which are destined to form halos are no different from those of randomly placed patches. However, halos have been shown to collapse around special positions in the initial density field [14] corresponding mostly to initial overdense ellipsoidal patch of matter [24]. Hence, even assuming an ellipsoidal collapse barrier, the model parameters for which the excursion set mass function reproduces that from N-body simulations may not be the same as those expected from the ellipsoidal collapse model. Most importantly, it is not *a pri-*

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ori guaranteed that the barrier model assumed in the excursion set calculation is also capable of reproducing the distribution of initial overdensities associated with N-body halos. In [10, 21, 22] the authors has proposed to implement a peak criteria into the excursion set theory prediction to get a physical insight¹. Alternatively one can still work within the excursion set and calibrate the collapse parameters such as to reproduce N-body simulation. In [17–20] the authors has shown that a diffusive drifting barrier (DDB), reproduces the mass function from N-body simulations to within $\sim 5\%$ accuracy.

In this Letter, we show that if one considers the statistics of all (rather than special) walks having to cross an effective boundary – one which may not be the same as that associated with the physics of collapse – then one can still build a self-consistent excursion set theory which reproduces with high accuracy N-body prediction of the halo mass function. We do so using halos identified with the Spherical Overdensity (SO) algorithm, with overdensity parameter $\Delta = 200$ in N-body simulations from the DEUS consortium [28] described in [29, 30]. The simulation box was $162h^{-1}$ Mpc and $648h^{-1}$ Mpc on a side with 512^3 particles, realized using the RAMSES code [23] for a Λ CDM model calibrated to WMAP-5yr data.

The excursion set approach assumes that $f(M)dM$ equals the fraction of walks which first exceed a critical value on a smoothing scale R corresponding to a mass M ($= \rho V(R)$). Since $\sigma(R)$ is a deterministic function of M , if one thinks of the walk height δ as being the sum of many steps, then the problem is to find the smallest σ at which

$$\delta \geq \delta_{th}, \quad (1)$$

where δ_{th} is a density threshold of collapse (i.e. barrier height). This may be a constant or vary with σ provided it has the same value at any point in space. Averaging over all possible walks is like averaging over all positions in space; this yields an estimate of the “first-crossing distribution”, $f(\sigma)d\sigma$. Since σ and M are related through R , one gets the excursion set estimate of the mass fraction from setting $f(M)dM \equiv f(\sigma)d\sigma$.

From the study of the ellipsoidal collapse it follows that the density threshold δ_{th} not only can vary with σ , but it may differ from point to point [12, 14, 25–27], thus exhibiting a stochastic behaviour. This motivates study of estimating the smallest σ for which

$$\delta \geq \delta_{th} \equiv \bar{B}(\sigma) + B, \quad (2)$$

with B independent of δ . We will assume that B , like δ , is a Gaussian variate, but with zero mean and variance $D_B\sigma^2$, while $\bar{B}(\sigma)$ is a deterministic function which

encapsulates the main feature of the ellipsoidal collapse. Note this model is very different from the one in [22] since the stochasticity depends explicitly of the scale we consider as expected from original ellipsoidal prediction [15, 31] after marginalising over ellipticity and prolativity of the ellipsoidal patch of matter in a Gaussian random density field. However the Gaussian distribution of the barrier does not correspond to the ellipsoidal prediction but as we mention before, since the excursion set theory assumes that proto halos form from randomly placed patches one might appeal to the central limit theorem and consider such model of the barrier or invoke that a Gaussian model is the simplest approximation one might consider to underline the stochastic behaviour of the barrier. Note also this double-Gaussian model turns out to have a number of other interesting properties: see [32] for details.

Written this way, one may think of the right-hand-side of Eq. (2) as being a stochastic barrier which δ must cross. While suggestive, this turns out to not be the best way to think of the problem, a point we will return to shortly. For example, the associated first crossing distribution is trivially solved by noting that $\delta - B$ is itself Gaussian with mean zero and variance $\sigma^2(1 + D_B)$, so this problem is, in fact, *exactly* the same as that of Eq. (1). In particular, the first crossing distribution will have the same functional form, except for a rescaling of $\sigma^2 \rightarrow \sigma^2(1 + D_B)$ as noticed by [6]. Note that, if both walks are smoothed with the same filter, then this is true whether or not the steps in each walk are correlated.

There is a sense in which this stochastic barrier model differs fundamentally from a deterministic one. Namely, the condition $\delta - B = \bar{B}(\sigma)$ may be satisfied at many different values of δ [15]. We will use δ_{1x} to denote the value of δ at first crossing, and will argue shortly that a combined comparison of $f(\sigma)$ and $\Pi(\delta_{1x}, \sigma)$ with their estimates from N-body simulations allows for a nice test of the excursion set approach.

In the special case where $\bar{B}(\sigma) = \delta_c + \beta\sigma^2$ (with δ_c the spherical collapse threshold), and the steps are uncorrelated, the first crossing distribution is [33]

$$f_0(\sigma) = \frac{\delta_c}{\sigma} \sqrt{\frac{2a}{\pi}} e^{-\frac{a}{2\sigma^2}(\delta_c + \beta\sigma^2)^2}, \quad (3)$$

where $a = 1/(1 + D_B)$. The corresponding distribution for tophat smoothing of both δ and B is discussed in Section 2.4 of [10].

Here instead, we will use tophat smoothing for δ but assume a Markovian random walk for B ; i.e., we assume that the collapse conditions at different scales are uncorrelated, and, to match scales, we assume that $\langle \delta^2 \rangle = \langle B^2 \rangle$. This requires that the scale of the sharp-k filter (which we use for B) must be about a factor of 2 smaller than that for the sharp-x filter (which we use for δ). Then we solve for the first crossing distribution by

¹ for realistic modelling of ellipsoidal peaks, this approach required a calibration of the collapse parameters using N-body simulation

walks in the variable $B - \delta$ with average $\delta_c + \beta S$ (as defined above) and variance $(1 + D_B)\min(S, S') + \Delta(S, S')$, where $\Delta(S, S')$ is a function that accounts for the sharp-x filter induced correlations in δ in order to have a coherent mass definition. Following [5] this is well approximated as $\Delta(S, S') = \kappa S/S'(S' - S)$ where $\kappa < 1$ is set by the linear matter power spectrum, and the first crossing distribution is

$$f(\sigma) = f_0(\sigma) + f_{1,\beta=0}^{m-m}(\sigma) + f_{1,\beta(1)}^{m-m}(\sigma) + f_{1,\beta(2)}^{m-m}(\sigma) \quad (4)$$

with

$$f_{1,\beta=0}^{m-m}(\sigma) = -\tilde{\kappa} \frac{\delta_c}{\sigma} \sqrt{\frac{2a}{\pi}} \left[e^{-\frac{a\delta_c^2}{2\sigma^2}} - \frac{1}{2} \Gamma\left(0, \frac{a\delta_c^2}{2\sigma^2}\right) \right], \quad (5)$$

$$f_{1,\beta(1)}^{m-m}(\sigma) = -a \delta_c \beta \left[\tilde{\kappa} \text{Erfc}\left(\delta_c \sqrt{\frac{a}{2\sigma^2}}\right) + f_{1,\beta=0}^{m-m}(\sigma) \right], \quad (6)$$

$$f_{1,\beta(2)}^{m-m}(\sigma) = -a \beta \left[\frac{\beta}{2} \sigma^2 f_{1,\beta=0}^{m-m}(\sigma) + \delta_c f_{1,\beta(1)}^{m-m}(\sigma) \right], \quad (7)$$

where $\tilde{\kappa} = a \kappa$ [18]².

For Λ CDM cosmology, $\delta_c = 1.673$ and $\kappa = 0.465$, so Eq. (4) has two free parameters: β and D_B . We determine these by fitting Eq. (4) to the DEUS halo counts, finding $\beta = 0.074$ and $D_B = 0.43$. Fig. 1 shows that, for this pair of values, the discrepancy between Eq. (4) and the numerical data at $z = 0$ is $\sim 5\%$, which is consistent with [17–19] and with numerical uncertainties on the mass function [30]. Fig. 1 also shows that Eq. (4) provides a similarly good description of the associated first crossing distribution, confirming [18], and illustrating that the excursion set approach can provide a good description of halo abundances.

We now turn to the question: Is this a self-consistent description? To answer this, we exploit the qualitatively new feature of stochastic barrier models: that walks which first-cross on scale σ will do so at a range of values of δ_{1x} . In the usual approach, $\Pi(\delta_{1x}, \sigma)$ is a delta function centered on δ_c .

To see what it is here, we begin by noting that in the coordinate system (g_1, g_2) where $g_2 = B/\sqrt{D_B}$ and $g_1 = \delta$, the crossing condition $\delta = \bar{B}(S) + B$ defines a line. It is in this sense that it is better to not think of the “barrier” to be crossed as being “stochastic”, but as a 2-dimensional random walk crossing a deterministic barrier (the line). Moreover the fact that this barrier is a line means that it is *much* more convenient to think of the

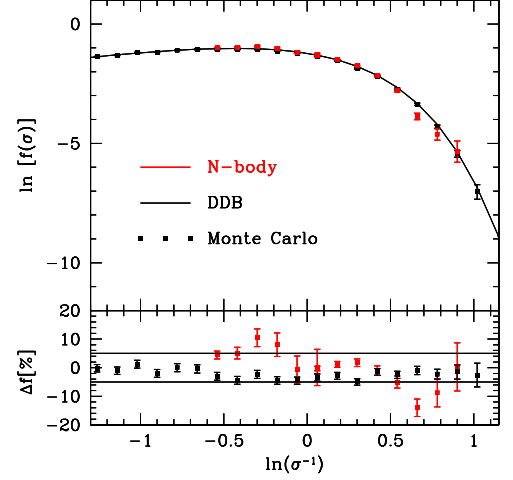


FIG. 1: (Upper panel) $f(\sigma)$ of the DDB model as given by Eq. (4) (black solid line) and the corresponding numerical solution from Monte-Carlo simulations (black squares) for $\beta = 0.074$ and $D_B = 0.43$. The red data points show the multiplicity function of SO(200) halos from N-body simulations. (Lower panel) Relative difference with respect to Eq. (4). The thin black solid lines indicates 5% deviations.

2-D walk as taking steps which are parallel and perpendicular to the barrier. Therefore, one should rotate the coordinate system to:

$$\begin{aligned} g_+ &= (g_1 - \sqrt{D_B} g_2) / \sqrt{(1 + D_B)} \\ g_- &= (g_1 \sqrt{D_B} + g_2) / \sqrt{1 + D_B} \end{aligned} \quad (8)$$

Notice that $\langle g_+ \rangle = \langle g_- \rangle = 0$, $\langle g_+^2 \rangle = \langle g_-^2 \rangle = \sigma^2$, and $\langle g_+ g_- \rangle = 0$. At $\delta_{1x} = \bar{B}(S) + B = \bar{B}(S) + g_2 \sqrt{D_B}$, we have

$$\begin{aligned} g_+ &= \bar{B}(S) / \sqrt{(1 + D_B)} \\ g_- &= \bar{B}(S) \sqrt{D_B} / (1 + D_B) + g_2 \sqrt{1 + D_B} \end{aligned} \quad (9)$$

making $g_2 = g_- / \sqrt{1 + D_B} - \bar{B}(S) \sqrt{D_B} / (1 + D_B)$ and hence

$$\delta_{1x} = \bar{B}(S) + g_2 \sqrt{D_B} = \frac{\bar{B}(S)}{1 + D_B} + g_- \sqrt{\frac{D_B}{1 + D_B}}. \quad (10)$$

Since g_- is a zero-mean Gaussian with variance σ^2 , the PDF of δ_{1x} at σ is that of g_- , with mean shifted by $\mu_{1x} \equiv \bar{B}(S) / (1 + D_B)$ and variance rescaled by $D_B / (1 + D_B)$, hence

$$\Pi(\delta_{1x}, S) = \frac{e^{-\frac{(\delta_{1x} - \mu_{1x})^2}{2SD_B^{\text{eff}}}}}{\sqrt{2\pi SD_B^{\text{eff}}}} \quad (11)$$

where $D_B^{\text{eff}} = D_B / (1 + D_B)$. The argument above is independent of the specific choice of $\bar{B}(S)$, so it is valid

² Eq. (7) includes a term $\mathcal{O}(\beta^2)$ corrected in [20].

for any value of the barrier, not just the spherical collapse case for which $\bar{B}(S) = \delta_c$. In the case of correlated walks due to smoothing δ with a sharp- x filter one may expect a correction to the above formula. However, since in this coordinate system there is no absorbing boundary associated with the random walk of g_- , then, the non-Markovian correction vanishes as shown in [5] and Eq. (11) remains valid also in this case.

We test the validity of this formula by comparing the distribution of values of δ_{1x} at a given σ measured from direct Monte Carlo simulations of the excursion set walks. In practice, we run walks with a barrier given by $\beta = 0.074$ and $D_B = 0.43$ (the values required for our first crossing distribution to fit the DEUS mass function), and we measure $\Pi(\delta_{1x}, \sigma)$ in $\ln(\sigma)$ with bins of width 0.1. The results are shown in Fig. 2 at $S = 1.5$, $S = 2$ and 3 . As we can see Eq. (11) describes the Monte Carlo measurements well.

If the excursion set approach is self-consistent, then $\Pi(\delta_{1x}, \sigma)$, with parameters calibrated from fitting $f(\sigma)$ to halo counts in simulations, should provide a good description of the distribution of δ measured in the N-body simulations. To estimate this in the DEUS simulations we select a random particle for each halo of mass M_{200} and evaluate the overdensity δ_{1x} within a sphere of radius $R = (3M_{200}/4\pi\rho)^{1/3}$ around it in the initial conditions. Choosing a random halo particle (rather than the one at the protohalo center, say) is crucial, since this makes the measurement correspond to the quantity which the usual excursion set calculation returns [35]: i.e., an average over all positions in the initial field. In contrast, the particle which lies closest to the initial center of mass represents a special subset of all walks [14]. We will consider such walks shortly.

Binning in $\sigma(M_{200})$ yields $\Pi(\delta_{1x}, \sigma)$ shown in Fig. 2. Our values of $S = 1.5$, $S = 2$ and 3 correspond to masses $M/10^{12}M_\odot = 22, 7.8$ and 2.2 (566, 1467 and 6185 halos, respectively). We do not have enough halos to test smaller values of S . Fig. 2 shows that $\Pi(\delta_{1x}, \sigma)$ for these halos is in good agreement with the excursion set predictions (analytical and numerical) for the barrier model parameters which best-fit the halo mass function suggesting an impressive self-consistency of the excursion set approach at low mass halos. Indeed, we measure a mean and a variance of distribution which are respectively $(1.277 \pm 0.033, 1.255 \pm 0.023, 1.314 \pm 0.015)$ and $(0.481 \pm 0.032, 0.627 \pm 0.026, 0.868 \pm 0.020)$ to be confronted to the mean and the variance of our theoretical prediction $(1.248, 1.275, 1.326)$ and $(0.448, 0.601, 0.894)$.

As already stressed, choosing a random particle from each halo, rather than the one at the center of mass is crucial. To see that this matters greatly, the (more sharply peaked) red histograms in Fig. 2 show δ_{1x} associated with center-of-mass particles in the same halos. The values associated with halo centers-of-mass tend to be larger and

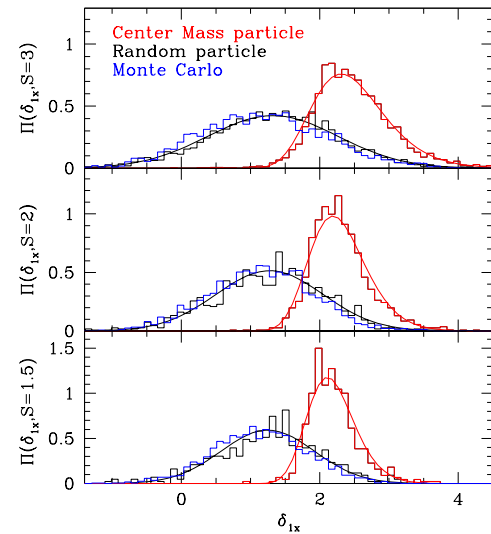


FIG. 2: Distribution of first-crossing overdensities in Monte-Carlo simulations (blue histograms) and our prediction Eq. (11) (smooth solid black curves), for parameters calibrated using the first crossing distribution shown in Fig. 1. and the initial overdensities around randomly chosen halo particles (black histograms) at $S = 1.5, 2$ and 3 . Red histograms, which are more sharply peaked, show the same measurement but around the halo centers of mass. In this case, smooth curves show the best-fitting Lognormal.

are almost always greater than δ_c . This is consistent with the analysis in [14], which implies that δ_{1x} for the center-of-mass particle in a halo will be larger than for any other particle in the halo. In addition, the distribution for center-of-mass particles is well described by a lognormal in agreement with previous work [36]. Over the range of mass considered we find

$$\Pi(\delta_{cm}, S) = \frac{e^{-\frac{(\ln[\delta_{cm}] - \mu_{cm})^2}{2D_{cm}S}}}{\delta_{cm} \sqrt{2\pi D_{cm}S}} \quad (12)$$

with $\mu_{cm} = \ln[(\delta_c + \beta_{cm}S^\gamma)/q] - D_{cm}S/2$ and $\{\beta_{cm} = 0.47, \gamma = 0.615, q = 1.04, D_{cm} = 0.0167\}$. At $S = 1.5, 2, 3$ this expression leads to a mean value $\langle\delta_{cm}\rangle = (2.188, 2.301, 2.492)$ while the N-body analysis gives $(2.186 \pm 0.355, 2.289 \pm 0.012, 2.486 \pm 0.009)$. This is in reasonable agreement with the crude estimate from the ellipsoidal collapse model of [14], suggesting that one might be able to build a physically consistent model for center of mass walks. However, the same crude averaging which led to the predicted mean relation suggests that the rms scatter around it should be $\sim 0.18\sigma$ while the data analysis leads to $(0.126 \pm 0.008, 0.175 \pm 0.007, 0.323 \pm 0.007)$ and Eq. (12) gives $(0.121, 0.179, 0.316)$ at $S = 1.5, 2, 3$. Some of the discrepancy may be due to the fact that our estimate of δ_{cm} is broadened as a consequence of having assumed the initial volume to be spherical, when it is usually not [37].

Although the excursion set approach does not predict the distribution around this mean for center-of-mass walks. This shortcoming would be resolved if we had an excursion set model for center-of-mass walks; this is the subject of work in progress. In this respect, one must view our results as indicating that the excursion set approach may provide a good effective model of the statistics of collapse, though its relation to the physics still remain unclear. Recently, [22] have argued that, within the context of peaks theory, one can build a model for the center-of-mass walks, from which $\Pi(\delta_{1x}, \sigma)$ can be predicted. Note that this approach also requires the scatter amplitude of the barrier to be calibrated with N-body simulations. One might think that for these type of walks the PDF of δ_{1x} matches the one of the center-of-mass walks. Unfortunately this consistency check has not been proved by the authors.

Alternatively, we can perform a similar study of the excursion set theory but for a log-normally distributed barrier and test whether the distribution of δ_{1x} recovers that of center-of-mass particles.

Overall, it is remarkable that our simple Gaussian model for the barrier yields a self-consistent excursion set approach and predict an analytic halo mass function with unprecedented agreement with respect to N-body simulation at the cost of only two collapse parameters which are coherent with the N-body analysis for the range of mass we investigate. Whether it is merely an effective approach or provides the stochastic master equation description of the gravitational N-body problem still remains an open question which we hope to address in the future.

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